Time-Frequency Analysis

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Indian Institute of Science

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Contents

Fourier Series as Basis Expansions

Fourier Basis for $\mathbb{C}^N$
(a.k.a discrete-time Fourier series/discrete Fourier transform/FFT)

The Fourier basis for $\mathbb{C}^N$ consists of the functions:

$$e_n(t) = \frac{1}{\sqrt{N}} e^{\mathrm{j}2\pi nt/N}, \quad n = 0, 1, \ldots, N-1$$

These functions are orthonormal:

$$\langle e_n, e_m \rangle = \frac{1}{N} \sum_{k=0}^{N-1} e^{\mathrm{j}2\pi(n-m)k/N} = \delta_{n-m}$$

Fourier basis for $L_2([-\pi,\pi])$

The Fourier basis for $L_2([-\pi,\pi])$ consists of the functions:

$$e_n(t) = \frac{1}{\sqrt{2\pi}} e^{\mathrm{j}nt}, \quad n \in \mathbb{Z}$$

These functions are orthonormal:

$$\langle e_n, e_m \rangle = \int_{-\pi}^{\pi} e_n(t)\overline{e_m(t)}\,\mathrm{d}t = \delta_{n-m}$$

The Fourier Transform on $L_1(\mathbb{R})$

The Fourier Transform on $L_1(\mathbb{R})$

Definition (Fourier transform in $L_1(\mathbb{R})$)

For a function f in the space $L_1(\mathbb{R})$, defined as:

$$L_1(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{C} : \int_{-\infty}^{\infty} |f(t)| \, \mathrm{d}t < \infty \right\}$$

The Fourier transform $\hat{f}$ (or $\mathcal{F}f$) is a function on $\mathbb{R}$ given by:

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-\mathrm{j}\omega t} \, \mathrm{d}t$$

Here, ω represents the angular frequency.

The Riemann-Lebesgue Lemma

Lemma

The Fourier transform maps $L_1(\mathbb{R}) \to C_0(\mathbb{R})$ *with the following properties*

- $\hat{f}$ *is a continuous function.*
- $\lim_{|\omega|\to\infty} \hat{f}(\omega) = 0.$
- *The transform is a bounded linear operator:* $\|\hat{f}\|_\infty \leq \|f\|_1$.

Properties of the L_1 Fourier Transform (I)

Let $f \in L_1(\mathbb{R})$.

Translation

For $a \in \mathbb{R}$, let $(\tau_a f)(t) = f(t-a)$.

$$\mathcal{F}(\tau_a f)(\omega) = e^{-\mathrm{j}\omega a}\hat{f}(\omega)$$

(Translation in time becomes a phase shift in frequency.)

Properties of the L_1 Fourier Transform (I)

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Modulation

For $\omega_0 \in \mathbb{R}$, let $(E_{\omega_0} f)(t) = e^{\mathrm{j}\omega_0 t} f(t)$.

$$\mathcal{F}(E_{\omega_0} f)(\omega) = \hat{f}(\omega - \omega_0)$$

(Modulation in time becomes a translation in frequency.)

Properties of the L_1 Fourier Transform (II)

Differentiation

$$\mathcal{F}(f')(\omega) = \mathrm{j}\omega \hat{f}(\omega)$$

Properties of the L_1 Fourier Transform (II)

Differentiation

If $f' \in L_1(\mathbb{R})$, then:

$$\mathcal{F}(f')(\omega) = \mathrm{j}\omega \hat{f}(\omega)$$

(Differentiation becomes multiplication by $\mathrm{j}\omega$.)

The Inverse Fourier Transform on $L_1(\mathbb{R})$

The Inversion Formula

Under suitable conditions, a function f can be recovered from its Fourier transform $\hat{f}$.

If both $f \in L_1(\mathbb{R})$ and its transform $\hat{f} \in L_1(\mathbb{R})$, then the inverse Fourier transform is given by:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{\mathrm{j}\omega t} \, \mathrm{d}\omega$$

- The condition that $\hat{f}$ must also be in L_1 is restrictive. For example, the transform of a rectangular pulse is a sinc function, which is not in L_1.

The Fourier Transform on $L_2(\mathbb{R})$

The Fourier Transform on $L_2(\mathbb{R})$

Theorem (Plancherel's Theorem)

The Fourier transform can be uniquely extended to a unitary operator on all of $\mathrm{L}_2(\mathbb{R})$.

- ***Isometry (Energy Preservation):***

$$\frac{1}{2\pi}\|\hat{f}\|_2^2 = \|f\|_2^2 \quad \textit{or} \quad \frac{1}{2\pi}\int_{-\infty}^{\infty}|\hat{f}(\omega)|^2\,\mathrm{d}\omega = \int_{-\infty}^{\infty}|f(t)|^2\,\mathrm{d}t$$

- ***Inner Product Preservation:***

$$\frac{1}{2\pi}\langle\hat{f},\hat{g}\rangle = \langle f,g\rangle$$

- *As a unitary operator, the L_2 Fourier transform is invertible, with $\mathcal{F}^{-1} = \mathcal{F}^*$.*

The Fourier Transform as a Distribution

Test Functions: The Schwartz Space $\mathcal{S}(\mathbb{R})$

Definition (Schwartz Space)

The **Schwartz space** $\mathcal{S}(\mathbb{R})$ is the set of all infinitely differentiable functions $\varphi : \mathbb{R} \to \mathbb{C}$ that, along with all their derivatives, decay faster than any polynomial.

$$\varphi \in \mathcal{S}(\mathbb{R}) \iff \sup_{t \in \mathbb{R}} \left| t^m \frac{\mathrm{d}^k}{\mathrm{d}t^k} \varphi(t) \right| < \infty \quad \forall k, m \in \mathbb{Z}_+$$

Think of these as extremely well-behaved functions, like Gaussians.

The Dual Space $\mathcal{S}'(\mathbb{R})$

Definition

A **distribution** is a continuous linear functional on the Schwartz space $\mathcal{S}$. The space of all such distributions is the dual space, denoted $\mathcal{S}'(\mathbb{R})$.
If $T \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$, we denote the action of T on φ by $\langle T, \varphi \rangle$.

Examples

1. **Regular Distributions:** Any function f of polynomial growth defines a distribution T_f

$$\langle T_f, \varphi \rangle = \int_{\mathbb{R}} f(t)\varphi(t)\,\mathrm{d}t$$

2. **Singular Distributions:**
   - Dirac Delta: $\langle \delta, \varphi \rangle = \varphi(0)$ Evaluation Functional
   - Derivative of Delta: $\langle \delta', \varphi \rangle = -\langle \delta, \varphi' \rangle = -\varphi'(0)$

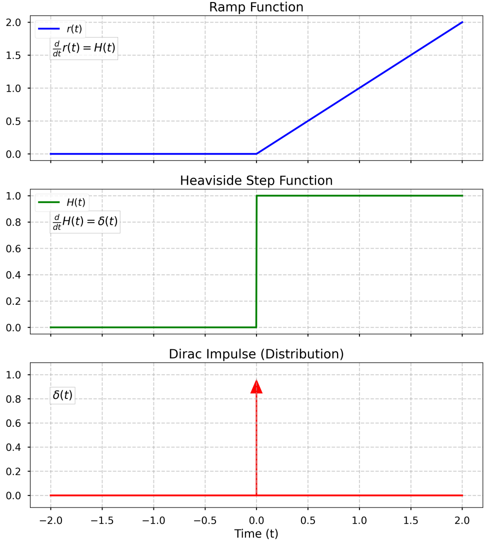
Operations on Distributions

- **Differentiation:** For $T \in \mathcal{S}'$, the derivative T' is defined by

$$\langle T', \varphi \rangle = -\langle T, \varphi' \rangle$$

- **Multiplication by Smooth Functions:** If $\psi \in \mathscr{C}^{\infty}(\mathbb{R})$,

$$\langle \psi T, \varphi \rangle = \langle T, \psi\varphi \rangle$$



The Fourier Transform of a Tempered Distribution

Definition by Duality

We extend the Fourier transform from $\mathcal{S}$ to $\mathcal{S}'$ using the duality pairing. For any $f, \varphi \in \mathcal{S}$, Plancherel's theorem implies:

$$\frac{1}{2\pi} \int \hat{f}(\omega)\varphi(\omega)\,\mathrm{d}\omega = \int f(t)\hat{\varphi}(t)\,\mathrm{d}t \implies \frac{1}{2\pi}\langle \hat{f}, \varphi\rangle = \langle f, \hat{\varphi}\rangle$$

This motivates the definition of the Fourier transform for any distribution $T \in \mathcal{S}'$:

$$\frac{1}{2\pi}\langle \hat{T}, \varphi\rangle = \langle T, \hat{\varphi}\rangle \quad \text{for all } \varphi \in \mathcal{S}$$

Examples of Distributional Fourier Transforms

Fourier Transform of the Dirac Delta

$$\frac{1}{2\pi}\langle\hat{\delta},\varphi\rangle = \langle\delta,\hat{\varphi}\rangle = \hat{\varphi}(0) = \frac{1}{2\pi}\int \varphi(t)\,\mathrm{d}t = \left\langle\frac{1}{2\pi},\varphi\right\rangle$$

Thus, the Fourier transform of the delta function is a constant:

$$\hat{\delta}(\omega) = 1$$

Fourier Representation of Dirac Delta

The inverse transform gives the famous (and purely formal) integral representation:

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\mathrm{j}\omega t}\,\mathrm{d}\omega$$

Fourier Transform of the Sine Function

A sine wave like $\sin(\omega_0 t)$ is a tempered distribution. Its Fourier transform is:

$$\sin(\omega_0 t) \quad \stackrel{\mathcal{F}}{\longmapsto} \quad -\mathrm{j}\pi \left(\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\right)$$

Convolution

Convolution of Functions

Definition

Let $f, g \in L_1(\mathbb{R})$, the **convolution** $f * g$ is given as:

$$(f * g)(t) = \int_{-\infty}^{\infty} f(t-\tau) g(\tau) \, \mathrm{d}\tau$$

Properties

If $f, g, h \in L_1(\mathbb{R})$:

- **Commutative:** $f * g = g * f$
- **Associative:** $(f * g) * h = f * (g * h)$
- **Young's Inequality:** $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$
  This ensures that if $f, g \in L_1$, then $f * g \in L_1$

The Convolution Theorem

Theorem

The Theorem For $f, g \in L_1(\mathbb{R})$*:*

$$\mathcal{F}\{f * g\}(\omega) = \hat{f}(\omega) \cdot \hat{g}(\omega)$$

Conversely, multiplication in time corresponds to convolution in frequency:

$$\mathcal{F}\{fg\}(\omega) = (\hat{f} * \hat{g})(\omega)$$

Convolution with Distributions

Definition

The concept of convolution can be extended to a test function $\varphi \in \mathcal{S}$ and a tempered distribution $T \in \mathcal{S}'$. The convolution $\varphi * T$ is a smooth function of polynomial growth:

$$(\varphi * T)(t) = \langle T, R\tau_t \varphi \rangle$$

where $\tau_t \varphi(\tau) = \varphi(\tau - t)$ is translation and $R\varphi(\tau) = \varphi(-\tau)$ is reflection.

Convolution with the Dirac Delta

The Dirac delta distribution acts as the identity element for convolution. For any $\varphi \in \mathcal{S}$:

$$(\varphi * \delta)(t) = \langle \delta, R\tau_t \varphi \rangle = (R\tau_t \varphi)(0) = \varphi(t)$$

Therefore:

$$\varphi * \delta = \varphi \quad \Longrightarrow \quad \int \delta(t - \tau)\varphi(\tau) d\tau = \varphi(t)$$

Linear and Time-Invariant Systems

Definition

A system T mapping input signals f to outputs $T(f)$ is:

- **Linear:** $T(af + bg) = aT(f) + bT(g)$ for all signals f, g and scalars a, b
- **Time-Invariant:** $T(\tau_a f) = \tau_a T(f)$

Impulse Response

The response of the system to the Dirac delta δ is called the impulse response h is defined as $h(t) = T(\delta)(t)$. Suppose φ is the input to a linear and time-invariant system, then the output is given by:

$$T(\varphi) = \varphi * h$$

Bandlimited Signals

Bandlimited Signals

Definition (Paley-Wiener space)

A function $f \in L_2(\mathbb{R})$ is bandlimited if its Fourier transform $\hat{f}$ has compact support. The Paley-Wiener space is the prototype bandlimited space, defined as

$$\mathrm{PW} = \left\{ f \in L_2(\mathbb{R}) : \text{supp } \hat{f} \subset [-\pi, \pi] \right\},$$

Properties of the Paley-Wiener Space

Lemma

If $f \in \mathrm{PW}$*, then* $\hat{f} \in L_1(\mathbb{R})$

Properties of the Paley-Wiener Space

Lemma

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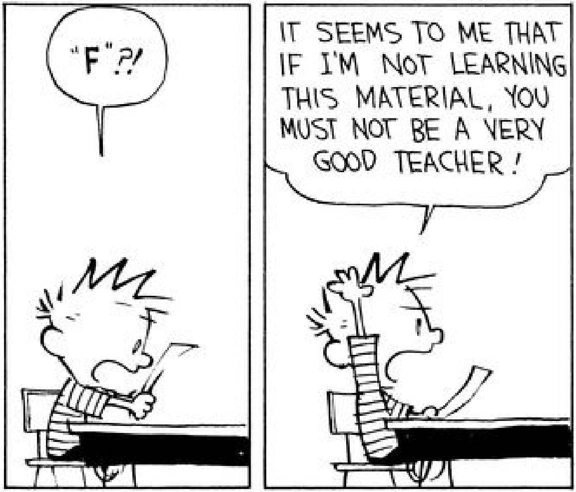
Theorem (Continuity of functions in PW)

Let $f \in \mathrm{PW}$*. Then,* f *is a continuous function.*


"F"?!
IT SEEMS TO ME THAT IF I'M NOT LEARNING THIS MATERIAL, YOU MUST NOT BE A VERY GOOD TEACHER!