Time-Frequency Analysis

E9 213 — August 2025
Indian Institute of Science

Course Instructor: Chandra Sekhar Seelamantula
TAs: Ram Rathan, Abijith Jagannath Kamath

August 5, 2025

Contents

A Problem With the Fourier Transform

The Fourier Transform

The Fourier transform of a signal $s(t)$ is defined as:

$$\hat{s}(\omega) = \int_{-\infty}^{\infty} s(t) e^{-i\omega t} \, \mathrm{d}t$$

Fourier Analysis is Limited

The Fourier transform tells us **what** frequencies are present in a signal, but it doesn't tell us **when** they occur.

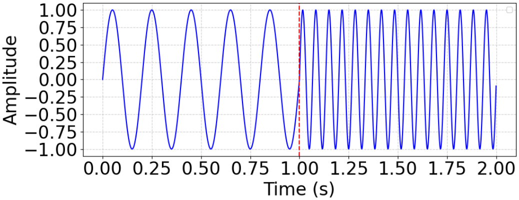
Design $P\{s\}(t, \omega)$ that simultaneously captures time and frequency.

An Example

Let's look at a non-stationary signal, $s(t)$, composed of piecewise sinusoids:

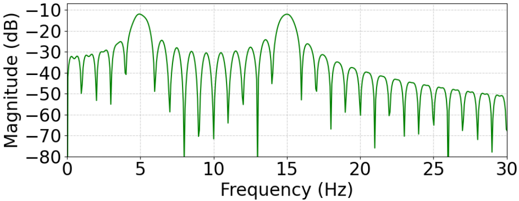
$$s(t) = \begin{cases} \sin(10\pi t) & 0 \leq t < 1 \\ \sin(30\pi t) & 1 \leq t \leq 2 \end{cases}$$

Time Domain View



Clearly shows *when* the signal changes. But what are the frequencies?

Frequency Domain View



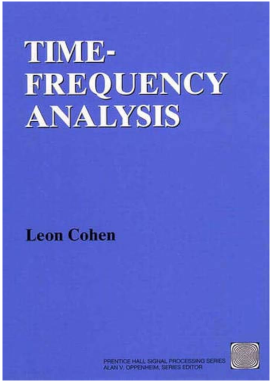
$|\hat{s}(\omega)|$ clearly shows two peaks at 5 Hz and 15 Hz. But when did they happen? We can't tell.

We need a simultaneous time-and-frequency transform

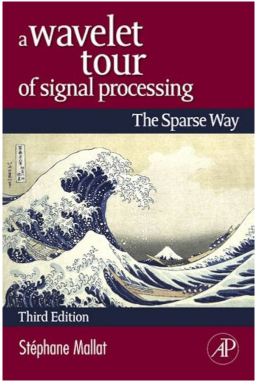
A Roadmap for the Course

1. **The Foundation:**
   - Hilbert Transform
   - Analytic Signals
   - Instantaneous Frequency
2. **The Linear Approach:**
   - The Short-Time Fourier Transform (STFT)
   - The Spectrogram
   - The Uncertainty Principle
3. **The Quadratic (Bilinear) Approach:**
   - The Wigner-Ville Distribution (WVD)
   - Cohen's Class of Distributions
4. **A Multi-Resolution Approach:**
   - Splines and Sampling Theorems
   - The Wavelet Transform

Primary Texts



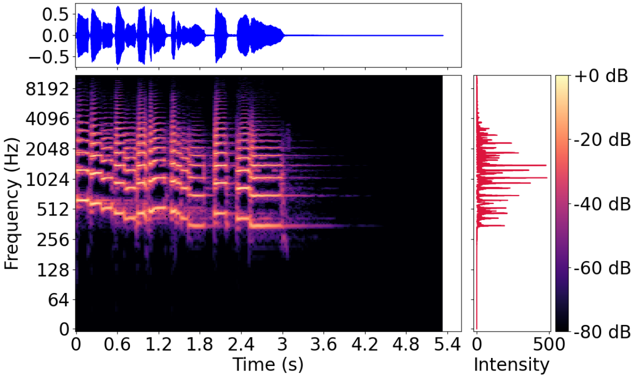
L. Cohen, *Time-Frequency Analysis*



S. Mallat, *A Wavelet Tour of Signal Processing*

Applications (I): Speech and Audio Processing

Speech/audio are typically non-stationary



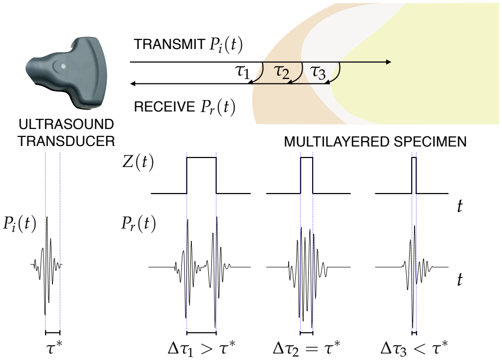
Spectrograms are often the inputs to neural networks for tasks such as speaker classification and recognition.

Ultrasound Imaging

- Image of the acoustic impedence of the specimen
- Received signal is a superposition of echoes from different depths

$$P_r(t) = \sum_{k=1}^{K} a_k P_i(t - \tau_k)$$

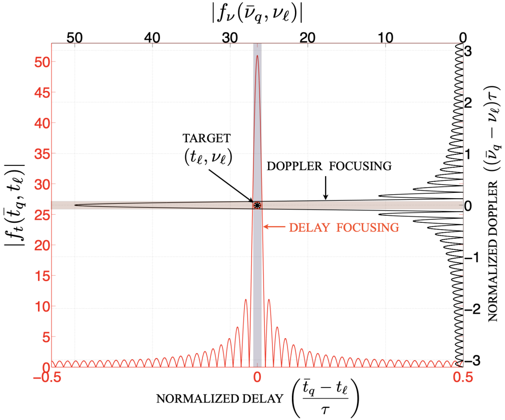
- Design pulses for higher resolution



Applications (III): Radar/Sonar

Radar/Sonar or Ranging

- Moving objects induce a Doppler shifts in frequency
- Signals are non-stationary!
- Change in frequency $\rightsquigarrow$ velocity estimation



Mathematical Preliminaries:
Normed Spaces, Banach Spaces and Hilbert Spaces

Vector Spaces (I)

Definition (Vector space)

A vector space over $\mathbb{C}$ is a set V with addition and multiplication that satisfies $\forall u, v, w \in V$ and $\alpha, \beta \in \mathbb{C}$

1. $v + w = w + v$
2. $\alpha(\beta v) = (\alpha\beta)v$
3. $(v + w) + u = v + (w + u)$
4. $(\alpha + \beta)v = \alpha v + \beta v$
5. $\alpha(v + w) = \alpha v + \alpha w$
6. $v + 0 = v$
7. $v + (-v) = 0$
8. $1v = v$

Vector Spaces (II): Subspaces

Definition (Subspace)

A subspace is a nonempty subset of a vector space that is *closed under addition and scalar multiplication*, i.e., $S \subseteq V$ is a subspace of V if $\forall v, w \in S$ and $\alpha \in \mathbb{C}$

1. $v + w \in S$
2. $\alpha x \in S$

Norms (I)

Definition (Norm)

A norm on a vector space V over $\mathbb{C}$ (or $\mathbb{R}$) is a real-valued function $\|\cdot\| : V \to \mathbb{R}$ with the following properties for any $v, w \in V$ and $\alpha \in \mathbb{C}$

1. $\|v\| \geq 0$ and $\|v\| = 0$ iff $v = 0$
2. $\|\alpha v\| = |\alpha| \|v\|$
3. $\|v + w\| \leq \|v\| + \|w\|$

A vector space endowed with a norm is called *normed vector space*.

Remark

1. $\|v - w\| \geq |\|v\| - \|w\||$
2. $\|v + w\|^2 + \|v - w\|^2 = 2(\|v\|^2 + \|w\|^2)$

Norms (II): Convergence

Definition (Convergence in normed spaces)

A sequence of vectors $(v_0, v_1, \cdots)$ in a normed vector space V is said to converge to $v \in V$ when $\lim_{k \to +\infty} \|v - v_k\| = 0$, i.e., given $\epsilon > 0$, there exists a $K = K(\epsilon)$ such that

$$\|v - v_k\| < \epsilon, \; \forall k > K.$$

Norms (III): Cauchy Sequences

Definition (Cauchy sequence)

A sequence of vectors $(v_0, v_1, \cdots)$ in a normed vector space is called a Cauchy sequence when given $\epsilon > 0$, there exists a $K = K(\epsilon)$ such that

$$\|v_k - v_m\| < \epsilon, \ \forall k, m > K.$$

Lemma (Convergent sequences are Cauchy)

Assume that V is a normed vector space, and that $(v_0, v_1, \cdots)$ is a convergent sequence in V. Then $(v_0, v_1, \cdots)$ is a Cauchy sequence.

Banach Spaces (I)

Definition (Banach space)

A normed vector space V with the property that each Cauchy sequence $(v_0, v_1, \cdots)$ in V converges toward some $v \in V$, is called a Banach space.

Banach Spaces (II): Examples

Examples:

- ℓ_p spaces

- L_p spaces

Inner Product Spaces (I)

Definition (Inner product space)

An inner product of a vector space V over $\mathbb{C}$ (or $\mathbb{R}$) is a complex-valued (or real-valued) function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ with the following properties for any $v, w, u \in V$ and $\alpha \in \mathbb{C}$ (or $\mathbb{R}$)

1. $\langle \alpha v + \beta w, u \rangle = \alpha \langle v, u \rangle + \beta \langle w, u \rangle$
2. $\langle v, w \rangle = \langle w, v \rangle^*$
3. $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ iff $v = 0$

Inner Product Spaces (II): Cauchy-Schwarz' Inequality

Theorem (Cauchy-Schwarz' inequality)

Let V be a vector space with an inner product $\langle \cdot, \cdot \rangle$. Then,

$$|\langle v, w \rangle| \leq \langle v, v \rangle^{1/2} \langle w, w \rangle^{1/2}, \ \forall v, w \in V.$$

Inner Product Spaces (III): Induced Norms

Lemma (Inner products induces the norm)

Let V be a vector space with an inner product $\langle \cdot, \cdot \rangle$. Then,

$$\|v\| = \langle v, v \rangle^{1/2}, \quad v \in V,$$

defines a norm on V.

Inner Product Spaces (IV): Hilbert Spaces

Definition (Hilbert space)

A vector space with an inner product $\langle \cdot, \cdot \rangle$, which is a Banach space with respect to $\| \cdot \| = \langle \cdot, \cdot \rangle^{1/2}$ is called a Hilbert space.

Inner Product Spaces (V): Examples

Examples:

- ℓ_2 space

- L_2 space

Inner Product Spaces (VI): Orthogonality

Definition (Orthogonality)

Let H be a Hilbert space.

1. Two elements $v, w \in H$ are *orthogonal* if $\langle v, w \rangle = 0$ and we write $v \perp w$
2. A collection of vectors $\{v_k\}_{k \in \mathbb{N}}$ in H is an *orthogonal system* if $\langle v_k, v_\ell \rangle = 0,\ \forall k \neq \ell$
3. An orthogonal system $\{v_k\}_{k \in \mathbb{N}}$ for which $\|v_k\| = 1,\ \forall k \in \mathbb{N}$ is called an *orthonormal system*

Bases, Orthogonal Bases, Riesz Bases

Basis (I)

Definition (Basis)

A set of vectors $\Phi = \{\varphi_k\}_{k \in \mathcal{K}} \subset V$, where $\mathcal{K}$ is countable, is called a *basis* for a normed vector space V when

- it is complete in V, i.e., for any $f \in V$, there exists a sequence $\alpha : \mathcal{K} \to \mathbb{C}$ such that

$$f = \sum_{k \in \mathcal{K}} \alpha_k \varphi_k,$$

- for any $f \in V$, the sequence α is unique.

Basis (II): Orthonormal Basis

Definition (Orthonormal Basis)

A set of vectors $\Phi = \{\varphi_k\}_{k \in \mathcal{K}} \subset H$, where $\mathcal{K}$ is countable, is called a *orthonormal basis* for the Hilbert space H when

- it is a basis for H, and
- it is an orthonormal set, i.e., $\langle \varphi_i, \varphi_k \rangle = \delta_{i-k} \; \forall i, k \in \mathcal{K}$.

Basis (III): Basis Expansion

Theorem (Orthogonal Basis Expansion)

Let $\Phi = \{\varphi_k\}_{k \in \mathcal{K}}$ be an orthonormal basis for a Hilbert spaces H. The unique expansion expansion coefficients for any $f \in H$ are given by

$$\alpha_k = \langle f, \varphi_k \rangle.$$

Synthesis with these coefficients yield

$$f = \sum_{k \in \mathcal{K}} \langle f, \varphi_k \rangle \varphi_k.$$

Parseval Equality

Theorem (Parseval Equality)

Let $\Phi = \{\varphi_k\}_{k\in\mathcal{K}}$ be an orthonormal basis for a Hilbert spaces H. The expansion coefficients satisfies the Parseval equality

$$\|f\|^2 = \sum_{k\in\mathcal{K}} |\langle f, \varphi_k\rangle|^2 = \|\alpha\|^2.$$

The generalised Parseval equality:

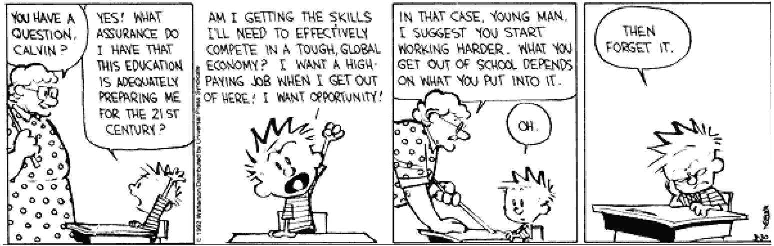
$$\langle f, g\rangle = \langle \alpha, \beta\rangle.$$

Basis (IV): Examples

Examples:

- Discrete Fourier basis for $\mathbb{C}^N$

- Fourier basis for $L_2([-\pi, \pi])$


YOU HAVE A QUESTION, CALVIN ?
YES! WHAT ASSURANCE DO I HAVE THAT THIS EDUCATION IS ADEQUATELY PREPARING ME FOR THE 21 ST CENTURY ?
AM I GETTING THE SKILLS I'LL NEED TO EFFECTIVELY COMPETE IN A TOUGH, GLOBAL ECONOMY ? I WANT A HIGH-PAYING JOB WHEN I GET OUT OF HERE! I WANT OPPORTUNITY!
IN THAT CASE, YOUNG MAN, I SUGGEST YOU START WORKING HARDER. WHAT YOU GET OUT OF SCHOOL DEPENDS ON WHAT YOU PUT INTO IT.
OH.
THEN FORGET IT.